

GKM - Homology of / family of equivariant ASF.
 off Spr fibers in the unramified case / Equivariant Cohomology of affine Springer fibers

blue = remarks for myself

I blatantly refer to my talk at the SERG reading group for basics on equivariant cohomology

$\gamma \in \mathfrak{t}(\mathfrak{g})$ r.s.
 G cpx reductive group.

$F = \mathbb{C}(\mathcal{E})$

$G = \mathbb{C}[[\mathcal{E}]]$

$Gr_G = G(F)/G(\mathfrak{g})$

$\gamma \in \mathfrak{t}(\mathfrak{g})$ regular semisimple. ("unramified case")

affine Springer fiber

$X_\gamma = \{ x \in Gr_G \mid \text{Ad}(x^{-1})\gamma \in \mathfrak{g}(\mathfrak{g}) \}$

finite dimensional ind-variety.

$T(\mathbb{C}) \curvearrowright X_\gamma : \quad z \in T(\mathbb{C}) \quad z \cdot g G(\mathfrak{g}) = (\text{Ad}(z)g) G(\mathfrak{g})$

Question : $H_*^{T(\mathbb{C})}(X_\gamma) = ?$

Rk: $\gamma = at^d \in \mathfrak{t}(F) \quad a \in \mathfrak{t}^{r.s.}$
 $d \geq 0$

$\lim_{d \rightarrow \infty} X_\gamma = X$

Answers to the question + limit gives $H_*^{T(\mathbb{C})}(X)$.

$\Lambda \subset T(F)$ lattice of translations

$$1 \rightarrow T(\omega) \rightarrow T(F) \xrightarrow{\text{val}} \mathcal{X}_*(T) \rightarrow 1$$

splitting (choosing uniformizing
parameter ε for $\mathcal{O}(\varepsilon)$)

w/ image Λ

$G \supset B \supset T \rightsquigarrow \phi^+$ positive roots

$$\alpha \in \phi^+$$

$$\alpha^\vee: F^x \rightarrow T(F) \rightsquigarrow \alpha^\vee \in \Lambda$$

$$\alpha^\vee \in t = t^{**} \rightsquigarrow \text{degree 1-monomial } \alpha_{\alpha^\vee} \in S(E)$$

$$\alpha \in t^* \simeq X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} \rightsquigarrow \text{degree 1-differential operatn } \partial_\alpha \in \mathcal{D}(E)$$

Sym(E)
" Sym(E)

" Sym(E*)

$$\partial_\alpha(\alpha_{\alpha^\vee}) = 2$$

$$S(E) \langle \partial_\alpha^d \rangle$$

submodule of pol funct. annihilated by ∂_α^d .

$$L_{\alpha, \gamma} = \sum_{d=1}^{\text{val}(\alpha^\vee(\gamma))} (1 - \alpha^\vee)^d \mathbb{C}[\Lambda] \otimes S(E) \langle \partial_\alpha^d \rangle$$

Chm: Suppose $H_*(X_\gamma; \mathbb{C})$ is pure. Then the inclusion

$\Lambda \subset X_\gamma$ induces an exact sequence

$$0 \rightarrow \sum_{\alpha \in \phi^+} L_{\alpha, \gamma} \rightarrow H_*^{T(\mathbb{C})}(\Lambda) \xrightarrow{\text{sr}} H_*^{T(\mathbb{C})}(X_\gamma) \rightarrow 0$$

$T(F)/T(O)$
 $\mathbb{C}[\Lambda] \otimes S(E)$

Fact GK17- Purity of equivariant Springer fibers
 purity holds for $\sigma = at^d$, $a \in t^{rss}$
 $d \geq 0$.

define affine paving of X_r by intersecting it w/ Schubert cells.

Limit

$$0 \rightarrow \bigoplus_{\alpha \in \phi^+} \sum_{d=1}^{\infty} (1-\alpha^d) \mathbb{C}[t] \otimes S(t) \{ \alpha^d \} \rightarrow H_*^{T(\mathbb{C})}(1) \rightarrow H_*^{T(\mathbb{C})}(X) \rightarrow 0$$

||
 "r

L_{∞} is preserved by $\tilde{W} \times \text{Aut}$.

So $\tilde{W} \times \text{Aut}$ acts on $H_*^{T(\mathbb{C})}(X)$.

A subgroup acts on the homology of affine Springer fibers:

$$(\tilde{W} \times \text{Aut})_{\sigma} = \left\{ \tau \in \tilde{W} \times \text{Aut} : \text{val}(\tau \alpha(\sigma)) = \text{val}(\alpha(r)) \right. \\ \left. \forall \alpha \in \phi^+ \right\}$$

preserves the module of relations $\sum_{\alpha \in \phi^+} L_{\alpha, r}$.

Homology of affine Springer fibers in the affine flag manifold.

$\Gamma \subset G(F)$ Iwahori

$$\begin{array}{ccc} \Gamma & \longrightarrow & B \\ \downarrow & & \downarrow \\ G(\mathcal{O}) & \longrightarrow & G \end{array}$$

$Y = Y^G = G(F) / \Gamma$ affine flag manifold

$\Lambda \subset T(F)$ lattice of translations.

$\alpha^\vee \in \phi^\vee(G, T) \sim \alpha^\vee(\mathcal{E}) \in \Lambda.$

$$\tilde{W} = \Lambda \rtimes W$$

$\alpha \in \phi^+$ give $\partial_\alpha \in \mathcal{D}(t)$ differential operators.

$\alpha^\vee \in \Lambda$ coroot

$w_\alpha \in W$ reflection

$$\begin{aligned} \text{Define } M_{\alpha, \gamma} &= \sum_{d=1}^{\text{val}(\alpha'(\gamma))} (1 - \alpha^\vee)^d \Phi[\tilde{W}] \otimes_{\mathbb{C}} S(t) \{ \partial_\alpha^d \} \\ &+ \sum_{d=1}^{\text{val}(\alpha'(\gamma))} (1 - \alpha^\vee)^{d-1} (1 - w_\alpha) \Phi[\tilde{W}] \otimes_{\mathbb{C}} S(t) \{ \partial_\alpha^d \}. \end{aligned}$$

$\gamma \in t(F)$

If G simply connected, Y^G classifies Iwahori subalgebras of $\mathfrak{g}(F)$

Y_γ those containing γ .

Thm: Let $\gamma \in t(G)$ be a regular element.

Suppose $H_*(Y_\gamma; \mathbb{C})$ is pure (implying formality)

Then the inclusion $\tilde{W} \subset Y_\gamma$ induces an exact sequence of $\mathcal{D}(t)$ -modules:

$$0 \rightarrow \sum_{\alpha \in \phi^\dagger} M_{\alpha, \gamma} \rightarrow \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} \mathcal{S}(t) \rightarrow H_*^{T(\mathbb{C})}(Y_\gamma) \rightarrow 0$$

$(\tilde{W} \rtimes \text{Aut})_\gamma \curvearrowright$ on $H_*^{T(\mathbb{C})}(Y_\gamma)$; it restricts to an action on the ordinary homology

$$H_*(Y_\gamma) = H_*^{T(\mathbb{C})}(Y_\gamma) \{ \mathbb{I} \}, \text{ subgroup of}$$

$H_*^{T(\mathbb{C})}(Y_\gamma)$ annihilated by the augmentation ideal

$$\mathbb{J} \subset \mathcal{D}.$$

$\text{Aut} =$ automorphism group of the based root datum for G

\subset Dynkin diagram automorphisms of G

$\overline{=}$ simply connected and adjoint cases

$$X_*(T) = \mathbb{Z}\phi^\vee$$

$$X^*(T) = \mathbb{Z}0$$

$\tilde{W} = \Lambda \rtimes W$ extended affine Weyl group

$\tilde{W} \rtimes \text{Aut} \curvearrowright$ on $T(\mathbb{F})$ and on the root system $\phi(G, T)$.
through $W \rtimes \text{Aut}$
 \curvearrowright on Λ

It acts diagonally on the left on

$$H_*^{T(\mathbb{F})}(\Lambda) \cong \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} S(t)$$

$(\tilde{W} \rtimes \text{Aut})_g$ passes acts on $H_*^{T(\mathbb{F})}(Y_g)$

and on $H_*(Y_g) = H_*^{T(\mathbb{F})}(Y_g)[\mathcal{I}]$
 \hookrightarrow aug. ideal of $\mathcal{D}(t)$.

Springer action (Mih-Tam talk)

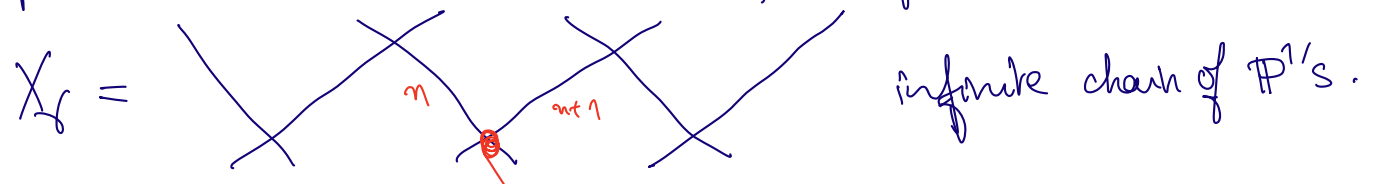
regular \otimes trivial $\curvearrowright \tilde{W}$ from the right on $\mathbb{C}[\tilde{W}] \otimes S(t)$.

It preserves each $H_{\alpha, \gamma}$ $\alpha \in \phi^+$

Passes to an action on $H_*^{T(\mathbb{F})}(Y_g)$ comp. w/ $\mathcal{D}(t)$ -module structure & commute w/ $(\tilde{W} \rtimes \text{Aut})_g$ -action.

↳ coincides w/ Lusztig's Springer action.

Example: $G = \text{SL}_2$, $\gamma = t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ equivariant Springer fiber



forms $T \simeq \mathbb{C}^* \subset G$ $t^n \mathbb{C}_F \oplus t^{-n} \mathbb{C}_F$ lattices

$\Lambda \subset \mathbb{C}((t)), \gamma \Lambda \subset \Lambda.$

$n \in \mathbb{Z}$

$C_n = \{ \Lambda \subset \mathbb{C}((t))^{\oplus 2} \mid$

$t^n \mathbb{C}[[t]] \oplus \mathbb{C} \subset \Lambda \subset t^{-n} \mathbb{C}[[t]] \oplus \mathbb{C}$
 $\left. \begin{array}{l} \underbrace{t^{-n+1} \mathbb{C}[[t]]}_{\Lambda'} \\ \underbrace{t^{-n} \mathbb{C}[[t]]}_{\frac{1}{t} \Lambda'} \end{array} \right\}$

Then $C_n \simeq \mathbb{P}^1$

T-action

$\tilde{u} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$

$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}$

$x \in t^{2n-1} \mathbb{C}_F / t^{2n} \mathbb{C}_F$

$\tilde{u} g \tilde{u}^{-1} = \begin{pmatrix} 1 & u^2 x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}$

so \mathbb{C}^* acts with weight 2 on each \mathbb{P}^1 component of X_γ with fixed points the intersection points $= 1 = \mathbb{Z}$

$\mathbb{C}[\Lambda] \otimes H_*^{\pi(\mathbb{C})}(\text{pt})$
 \downarrow
 $\mathbb{S}1 \quad \mathbb{C}[[x]]$

$$H_x^{\tau(\mathbb{C})}(\mathbb{Z}) \rightarrow H_x^{\tau(\mathbb{C})}(X_\gamma) \rightarrow 0$$

- $\alpha \in \phi^+$ unique positive root.

$$\text{val}(\gamma) = 1 \in \mathbb{Z}$$

- $L_{\alpha, \gamma} = (1 - d^\vee) \mathbb{C}[\Lambda] \otimes \mathbb{C}[x] \{ \partial_\alpha \}$

= pd functions annihilated by

∂

$\cong \mathbb{C}$ constant polynomial functions.

So that

$$H_x^{\tau(\mathbb{C})}(X_\gamma) \cong \frac{\mathbb{C}[\Lambda] \otimes \mathbb{C}[x]}{(1 - d^\vee) \mathbb{C}[\Lambda]}$$

① Purity

X cpx proj variety

Deligne: increasing weight filtration (W^m) on $H^*(X; \mathbb{C})$
 \Rightarrow increasing weight filtration on $H_*(X; \mathbb{C})$ by duality

" X pure" means by definition

$$\text{Gr}^m W(H^i(X)) = 0 \text{ if } m \neq i$$

$$\Leftrightarrow \text{Gr}^{-m} W(H_i(X)) = 0 \text{ if } m \neq i.$$

The strictness: If $f: X \rightarrow Y$ morphism of projective varieties,
 $f_*: H_*(X) \rightarrow H_*(Y)$ is strict w.r.t. the weight filtration:

$$(\text{im } f_*) \cap W^m(H^*(Y)) = f_* W^m(H^*(X)).$$

allows to extend the definition of the weight filtration
to any ind projective variety (inductive limit of projective
varieties) $X = \varinjlim_{i \in I} X_i$ by setting

$$W^m H_*(X) = \varinjlim W^m(H_*(X_i))$$

$$\hookrightarrow \varinjlim H_*(X_i) =: H_*(X).$$

[Homology commutes w/ direct limits]

[Rk : stickiness of differential]

Affine Springer fibers are ind-varieties
 \Rightarrow homology/cohomology spaces can be infinite dimensional.

It's good to see them as module over an as big as possible algebra to get a good grasp on them.

(even better if we know (a part of) the structure of the category of modules over this algebra.)

Here: Cherednik algebras. Subject of talk 7 by Sasha.
(graded)

② Equivariant homology

A cpx torus

$\alpha = \text{lie } A$

$$\chi^*(A) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \alpha^*$$

$$S(\alpha) = \text{Sym}(\alpha) = \bigoplus_{d=0}^{\infty} \text{Sym}^d(\alpha^*)^*$$

Cpx valued pol-functions on α^* .

$\varphi: A \rightarrow \mathbb{C}^*$ character. \rightarrow line bundle on BA \mathcal{L}_φ .
w/ 1st Chern class $c_1(\varphi)$

$$\mathcal{D}(\alpha) \xrightarrow{\sim} H^*(BA) = H_A^*(pt) \quad [\text{Chern-Weil iso}]$$

$A \times A \rightarrow A$ multiplication induces $m: BA \times BA \rightarrow BA$.
endows $H_*(BA)$ w/ algebra structure.

$$\begin{matrix} H_*^A \\ \parallel \\ H_* \end{matrix} (pt)$$

cap product $H_*^A(pt) \otimes H_A^*(pt) \rightarrow H_*^A(pt)$
module over \rightarrow

Hopf formula

$$x \in H^2(BA) \quad \left\langle \begin{matrix} \uparrow \\ c_1(\mathcal{L}) \end{matrix} \right\rangle$$

$m^* \mathcal{L} \ni$ a line bundle
over $A \times A$ associated to the
character $\varphi \circ m$ whose

derivative at 1 is

$$A \times A \rightarrow \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m \quad d\varphi_{(1,1)}$$

$$\begin{aligned} \varphi((1+h)(1+h')) &= \varphi(1+h+h') \\ &= 1 + d\varphi_1(h+h') \end{aligned}$$

So $m^* \alpha = \alpha \otimes 1 + 1 \otimes \alpha$

So α acts on $H_x^A(pt)$ as a derivation:

$$(\alpha \cap m(y, z)) = \alpha \cap m_*(y \otimes z)$$

$$= m_* (m^*(\alpha) \cap (y \otimes z))$$

$$\left(\begin{aligned} f_* (f^* \alpha \cap \sigma) \\ = \alpha \cap f_* \sigma \end{aligned} \right)$$

(naturality of cap-product)

$$= m_* (\alpha \otimes 1 \cap (y \otimes z) + (1 \otimes \alpha) \cap (y \otimes z))$$

$$= m_* (\alpha \cap y) \cap z + m_* (y \cap \alpha) \cap z$$

$$= m(\alpha \cap y, z) + m(y, \alpha \cap z)$$

inverse

$S(\alpha) \rightarrow H_x^A(pt)$ dual to Chern Weil

$$\begin{array}{ccc} \mathbb{Z}_k \otimes S_j & \rightarrow & S_{j-k} \\ \downarrow S & \searrow G & \downarrow S \\ H_A^{2k}(pt) \otimes H_A^j(pt) & \xrightarrow{\cap} & H_A^{j-2k}(pt) \end{array}$$

Change of torus

TCA

BT \rightarrow BA

$$I = \ker \left(H_A^*(pt) \rightarrow H_T^*(pt) \right)$$

$$H_T^*(X) = H_A^*(X) \otimes_{H_A^*(pt)} H_T^*(pt) = \frac{H_A^*(X)}{IH_A^*(X)}$$

[purity of $X \Rightarrow X$ equivariantly formal for any torus acting]

$$H_V^*(X) = H^*(X) \otimes H_V^*(pt).$$

$H_*^A(pt)$ is a module over $\mathcal{D} = H_A^*(pt)$

and $H_*^T(X) = H_*^A(X) \setminus \{I\}$ elements of $H_*^A(X)$ annihilated by the homogeneous ideal I
by $H_A^*(pt)$ -duality.

• A character thing: $\varphi: A \rightarrow \mathbb{C}^\times$
 $\partial\varphi \in \mathfrak{m}^* \subset \mathcal{D}(\mathfrak{m})$
 $\mathfrak{m} = \ker \mathcal{D} \subset \mathfrak{m}$

$j: pt/\mathfrak{n} \rightarrow pt/A$ induces

pull-back $\mathcal{D}(\mathfrak{m}) \rightarrow \mathcal{D}(\mathfrak{m})$ w/ kernel $\partial\varphi$.

Dualizing, we get $j^*: S(\mathfrak{m}) \rightarrow S(\mathfrak{m})$ w/ image polynomial annihilated by $\partial\varphi$.

extend the A-action to \mathbb{P}^1 .

$$0 \rightarrow H_*^A(\mathbb{C}\mathbb{P}^1, \{0\} \cup \{\infty\}) \xrightarrow{\partial} H_*^A(\{0\}) \oplus H_*^A(\{\infty\}) \rightarrow H_*^A(\mathbb{C}\mathbb{P}^1) \rightarrow 0.$$

$$\begin{array}{ccc} \parallel & & \parallel \\ S(m) & \xrightarrow{\beta} & S(\alpha) \oplus S(\alpha) \end{array}$$

$$f \longmapsto (j^*(f), -j^*(f)).$$

use that $\text{Ker } \partial$ is a subtorus of A (and then that $H_*^A(\mathbb{C}^*) \cong S(m^*)$).

[instead it's not true? or the only possible problem comes from torsion?]

$$H_*^A(\mathbb{C}\mathbb{P}^1, \{0\} \cup \{\infty\}) = H_*^A(\mathbb{C}^*) \cong S(m^*)$$

At the level of singular chains

dist. triangle

$$\text{pair } C_*^A(\{0\} \cup \{\infty\}) \rightarrow C_*^A(\mathbb{P}^1) \rightarrow C_*^A(\mathbb{P}^1, \{0, \infty\})$$

\downarrow iso

\parallel

\uparrow iso.

$$\text{Mayer-Vietoris } C_*^A(\mathbb{P}^1, \{0\}) \sqcup C_*^A(\mathbb{P}^1, \{\infty\}) \rightarrow C_*^A(\mathbb{P}^1) \rightarrow C_*^A(\mathbb{P}^1, \{0, \infty\}) \quad [1]$$

$C_*^A(\mathbb{P}^1, \{0, \infty\})$ pushforward for maps is dual to pullback in cohomology explaining the formula

+ connection morphism

Lemma (Chang - Skjelbred, homological version)

$A \curvearrowright Y$ cpx proj variety
 (pure)

$Y_0 \subset Y$ fixed points

$Y_1 \subset Y$ union of 0 & 1 dim orbits (= 1-dim skeleton)

$$H_*^A(Y_1, Y_0) \xrightarrow{\cong} H_*^A(Y_0) \longrightarrow H_*^A(Y) \longrightarrow 0 \text{ is exact.}$$

If $Y_0 = \{y_1, \dots, y_r\}$ finite

$Y_1 = \{E_1, \dots, E_d\}$ finite no.

$E_i = \{y_{i,a}\} \cup \{y_{i,b}\}$

$m_i = \dim(\text{Stab } E_i)$

$$j_i^* : S(m_i) \rightarrow S(\infty)$$

Complex obtained from the 1-D skeleton.

$$\bigoplus_{i=1}^d S(m_i) \xrightarrow{\beta} \bigoplus_{k=1}^r S(\infty) \longrightarrow H_*^A(Y) \longrightarrow 0$$

$$\beta_i(f_i) = (0 \dots 0 \ j_i^* f_i, \rightarrow -j_i^*(f_i) \cdot 0 \dots 0)$$

$$\beta = \sum \beta_i$$



bunch of \mathbb{P}^1 's - Reproduce the previous reasoning to this situation.

Chm (GKM - localization of equivariant cohomology)
 SERG Reading group Talks 2
 + homology version

Co-homology version

Under mild assumptions, $H_T^*(X)$ is determined by fixed points and 1-dimensional orbits.

X cpx proj var
 \uparrow
 $T \simeq (\mathbb{C}^*)^n$ finite number of fixed points, finite number of 1-dim orbits
 X equivariantly formal
 $H_T^*(X) \xrightarrow{L^*} H_T^*(X^T) \simeq \bigoplus_{x \in X^T} S(t^*)$ is injective
 E_j w/ $\bar{E}_j = E_j \cup \{x_{j^0}, x_{j^{n_0}}\}$

w/ image

$$H = \left\{ (f_1, \dots, f_k) \in H_T^*(X^T) \mid f_{j^0}/l_j = f_{j^{n_0}}/l_j \quad \forall 1 \leq j \leq k \right\}$$

$$l_j = \text{Ker} \left[\prod_{\chi \in \hat{T}} \chi \mid T \rightarrow \mathbb{C}^* \right]$$

character of the T -action on \bar{E}_j

superfluous.

Affine Springer fibers

$$X_*(T)$$

dual torus $\hat{T} = \text{Hom}(X_*(T), \mathbb{G}_m)$

$$T \subset G, \quad \hat{G} = (X_*(T), \phi^\vee, X^*(T), \phi).$$

$$(X^*(T), \phi, X_*(T), \phi^\vee)$$

$$F = \mathbb{C}((\varepsilon))$$

$$O = \mathbb{C}[[\varepsilon]] \text{ ring of integers.}$$

Valuation

$$T \text{ } \mathbb{C}\text{-torus.}$$

$$1 \rightarrow T(O) \rightarrow T(F) \xrightarrow{\text{val}} X_*(T) \rightarrow 1$$

is exact,

$$\alpha(\text{val}(e)) = \text{val}(\alpha(e))$$

$$\alpha \in X^*(T) \quad \bigcap_{F^x}$$

uniformizing parameter ε gives splitting

$$\mathbb{Z}^n \cong X_*(T) \rightarrow T(F)$$

$$(\beta_1, \dots, \beta_n) = \beta \mapsto \beta(\varepsilon)$$

$$\mathbb{G}_m \rightarrow T$$

$$z \mapsto (z^{\beta_1}, \dots, z^{\beta_n})$$

$$(\varepsilon^{\beta_1}, \dots, \varepsilon^{\beta_n})$$

image = "lattice of translations"

free abelian group of rk $\dim T$

Affine Springer fiber

G conn red / \mathbb{C}

$\text{Lie } G = \mathfrak{g}$

$$\mathfrak{g}(F) = \mathfrak{g} \oplus F$$

$\mathfrak{g}(0)$

$$K = G(0)$$

$X = G(F)/G(0)$ affine Grassmannian

ind algebraic variety.

$H \subset G$ conn red alg subgroup

$$H(K)/H(0) \hookrightarrow X$$

$$\left(\begin{array}{l} \text{Key fact} \\ G(0) \cap H(K) = H(0) \end{array} \right)$$

$\gamma \in \mathfrak{g}(F)$ gives rise to the affine Springer fiber

$$X_\gamma = \{x \in G(F)/K \mid \text{Ad}(x^{-1})(\gamma) \in \mathfrak{g}(0)\}$$

γ "compact" if $X_\gamma \neq \emptyset$.

KL88: $\gamma \in \mathfrak{g}(F)$ is rss iff X_γ is finite dimensional ind-subvariety of X .

Purity conjecture: If $\gamma \in \sigma_\gamma(F)$ is compact, regular, semisimple then $(H^i, H^i(X_\gamma; \mathbb{C}))$ is pure of weight i .

Purity of equisized affine Springer fibers GK 2003

$k = \bar{k}$, G conn red/ k

A max torus

$$\mathfrak{a} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$$

$$F = k((\epsilon)) \supset \mathfrak{o}$$

\bar{F}

$$\underline{G} = G(F)$$

$\gamma \in \mathfrak{a}$ a single apartment $\underline{\sigma}_\gamma = \sigma_\gamma(F)$

$\underline{\mathcal{G}}_\gamma, \underline{\mathcal{I}}_\gamma$ connected parabolic subgroup/algebra.

$\bar{\Phi} \subset X^*(A)$ roots of G

$\Phi = \{\bar{\alpha} + n \mid \bar{\alpha} \in \bar{\Phi}, n \in \mathbb{Z}\}$ affine roots

$\bar{\alpha} \in \bar{\Phi}$ gives $L_{\bar{\alpha}}: \mathfrak{k}_{\bar{\alpha}} \xrightarrow{\sim} \mathfrak{u}_{\bar{\alpha}} \subset \text{Lie } G$.

$\alpha = \bar{\alpha} + n$ affine root $\sim \mathfrak{g}_\alpha = L_{\bar{\alpha}}(t^n \mathfrak{k}) \subset \mathfrak{g}$.

$$\mathfrak{g}_{\alpha + \mathbb{Z}_{\geq 0}} := L_{\bar{\alpha}}(t^n G)$$

$\gamma \in \mathfrak{a} \rightsquigarrow \underline{\sigma}_{\gamma} = \text{sublie alg gen by } \{ \mathfrak{g}_{\alpha + \mathbb{Z}_{\geq 0}}, \alpha \in \Phi, \alpha(\gamma) \geq 0 \}$

Similarly for $\underline{\mathcal{G}}_\gamma$:

$\mathcal{F}_\gamma := \underline{\mathcal{G}} / \underline{\mathcal{G}}_\gamma$. $\gamma = 0$: affine Grassmannian

$\mu \in \mathfrak{g}$ determines a closed subset

$$\mathcal{F}_y(\mu) := \{ g \in \underline{G}/\underline{G}_y \mid \text{Ad}(g^{-1})(\mu) \in \mathfrak{g}_y \}$$

"offine Springer fiber"

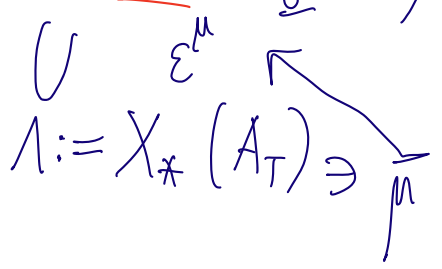
T max F -torus in G , \mathfrak{t}

A_T max split

$\mu \in \mathfrak{t}$ "integral" if $\text{val}(\lambda(\mu)) \geq 0 \quad \forall \lambda \in X^*(T)$

regular

$$T(F) = \underline{Z}_{\underline{G}}(\mu) \cong \mathcal{F}_y(\mu)$$



λ acts freely on $\mathcal{F}_y(\mu)$

$kL : \mathcal{F}_y(\mu)$ is projective / k , $\neq \emptyset \Leftrightarrow \mu$ integral, + formula dim. conjectured (proved by Beze...)

normalization: $\text{val}(\epsilon) = 1$

$\mu \in \mathfrak{t}^{\text{reg}}$ is equivalent with valuations $s \in \mathbb{Q}$ if

$\text{val } \alpha(\mu) = s \quad \forall \alpha$ root of T over \bar{F} and

$$\text{val}(\lambda(\mu)) \geq s \quad \forall \lambda \in X^*(T).$$

redundant for adjoint groups

Thm: Assume $\#W \in k^X$.
 absolute Weyl group of G .

μ regular equivalent of \mathfrak{t} . Then $\mathcal{F}_y(\mu)$ admits a Hessenberg variety

Hessenberg pairing: X ind. scheme

$X_0 \subset X_1 \subset \dots$ exhausting filtration

$X_i \setminus X_{i-1}$ disjoint union of iterated aff. space bundles over Hessenberg varieties.

$G(F)$ -conj. classes of max F -tori in G are par. by conj. classes in W .

T Coxeter if the corresponding conjugacy class consists of Coxeter elements
= product of all simple reflections.

$\Gamma = Z_G(A_T)$ is Levi subgroup of G , split / F .

\bigcup_T max F -tori

T "weakly Coxeter" if Coxeter in Γ .

split max tori are weakly Coxeter; all max tori in G_{an} are weakly Coxeter.

Hessenberg pairing obtained by intersecting off Springer fibers w/ orbits of parabolic subgroup depending on T .

If parabolic is Iwahori (e.g. T weakly Coxeter), resembles

- CL88 for ordinary Springer fibers

- coincide w/ LSG1, homogeneous μ ET, T Coxeter torus in G_{an} .

Chm: $\#W \in k^X$, Tweaked Coxeter.
 $u \in \text{Et reg integral equivalent.}$
 $\mathcal{F}_y(u)$ admits a paving by affine spaces.

weaker than
 "homogeneity"
 ↳ guarantees that
 loop rotation
 preserves the
 Springer fiber.

Not Gr: many pavings by non-affines

app Bk of KL: one ircomp of SF of a hum elt of
 $sp(6)$ has dom map to all curves.

More general Springer fibers

(affine)

$u \in \mathfrak{g}$ replaced by $v \in \underline{V} := V \otimes_{\mathbb{C}} F$, V f -dim \mathfrak{sp} of G over k .

\mathfrak{g}_y played by lattices $\underline{V}_{y,t}$ in \underline{V} , $t \in \mathbb{R}$, analogous to
 Meyer-Rand lattices in \mathfrak{g} .

→ generalized ASF $\mathcal{F}_y(t, v)$.

applications in GK unramified case

Laumon lemme fondamental p-les groupes unitaires.

TBC

Bruhat decomposition : $T \subset G$ max torus / \mathbb{F} (split over \mathbb{F})

$\Lambda \subset \Gamma(\mathbb{F})$ translation lattice

$$\cong \Gamma(\mathbb{F}) / \Gamma(\mathfrak{o})$$

$$\phi: \Lambda \rightarrow G(\mathbb{F}) / G(\mathfrak{o})$$

$B \supset T$ Borel; $I \subset G(\mathbb{F})$ Iwahori

$$G(\mathbb{F}) = I \Lambda K$$

$$\begin{array}{ccc} I & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ G(\mathfrak{o}) & \longrightarrow & G \\ & & \text{evaluation at } \varepsilon = 0. \end{array}$$

$$\Rightarrow X = \bigsqcup_{l \in \Lambda} I l K / K$$

off base

for $G = GL_n$, reduction of matrices of $\mathbb{C}[[\varepsilon]]$ principal unig.

$$\alpha_0 = K / K \subset X$$

$$C_l = I l \alpha_0 \subset X \text{ cell.}$$

loop rotation: $\Gamma_m \subset \mathbb{Q} \subset \mathbb{F}$ preserving G .

$$G^{\Gamma_m} = \mathbb{F} \cdot \varepsilon^0.$$

"

$$\mathbb{F}^{\Gamma_m}$$

$$\Gamma(\mathbb{C}) \subset I.$$

$\Gamma_m \subset X$ commutes w/ Γ -action.

Extended torus: $\tilde{T}(\mathbb{C}) = T(\mathbb{C}) \times \mathbb{C}^\times$ preserves the Loop rotations

Bruhat decomposition. \swarrow translation lattice

$$(t, \lambda) \cdot g \cdot k = \lambda (t g t^{-1}) k$$

since $T(\mathbb{C}) \cong G(F)/G(\mathcal{O})$ by conjugation,

Affine roots: $\phi = \phi(G, T) \supset \phi^+$ \mathfrak{b} Borel.

$$\mathfrak{a}_\phi(\mathbb{C}) = \mathfrak{t}(\mathbb{C}) \oplus \bigoplus_{\alpha \in \phi} \mathbb{C} Y_\alpha$$

root space dec.

$$\tilde{\phi} = \left\{ (\alpha, k) : \alpha \in \phi, k \in \mathbb{Z} \right\} \text{ affine roots.}$$

\swarrow
 induces character of $\tilde{T}(\mathbb{C})$ by $(\alpha, k)(t, \lambda) = \alpha(t) \lambda^k$.

$\tilde{T}(\mathbb{C})$ acts on the affine root space $\mathbb{C} \varepsilon^k Y_\alpha$ through this character.

\mathcal{C}_0 fundamental alcove determined by I

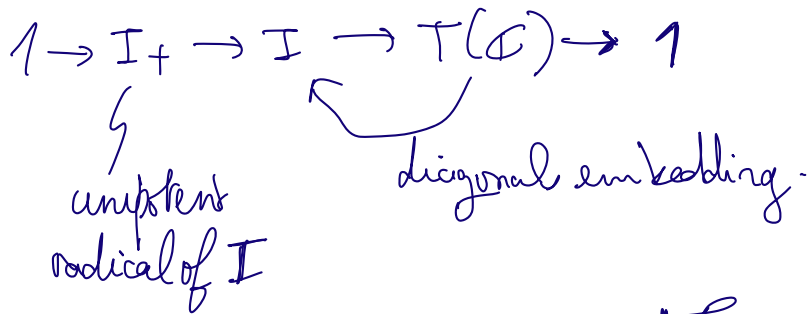
$\tilde{\Delta}$ simple affine roots = $\{\alpha_1, \dots, \alpha_n\} \cup \{\alpha_0\}$
 $\alpha_0 = -\theta + \delta$

$\theta =$ longest positive root of Δ

$$\mathcal{C}_0 = \left\{ a \in \mathcal{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{R} : \alpha(a) + k > 0 \quad \forall (\alpha, k) \in \tilde{\Delta} \right\}$$

\int
 $\mathfrak{h} \in \mathfrak{Tr}$

Split s.e.s.i



(picture for SL_2)

$$\mathfrak{K}(I) = \text{lie}(I_+) = \prod_{n \in \mathbb{Z} > 0} \mathbb{C} \varepsilon^n t(\mathbb{C}) \oplus \prod_{\substack{(\alpha, k) \in \check{\Phi} \\ \alpha(a) + k > 0 \\ \forall a \in \mathbb{C}_0}} \mathbb{C} \varepsilon^k Y_\alpha$$

(mistakenly 5.6.2 of GK12)

Lemma (Bruhat cells)

$$x_0 = k/k \in X$$

$$l \in \Lambda$$

exp map : $C_l = Ilx_0 \cong D_l := \bigoplus_{\substack{(\alpha, k) \in \check{\Phi} \\ \text{rel}(\alpha/k) + k < 0 \\ \alpha(a) + k > 0 \forall a \in \mathbb{C}_0}} \mathbb{C} \varepsilon^k Y_\alpha$

- * $T(\mathbb{C})$ -equiv obviously
- * exp commutes w/ loop rotation.
- * $T(\mathbb{C})$ -equivariant isomorphism.

finite set

(*)

Proof: analogous to the classical Bruhat decomposition.

I_+ acts transitively on C_l (since $T(\mathbb{C}) \subset I$ commutes with Λ)

$l x_0$ is stabilized by $I_+ \cap l G(l_0) l^{-1}$ whose lie algebra is

$$\left(g l k = l k \Leftrightarrow l g l \in \mathfrak{K} \right) \quad \underline{\text{connected}}$$

sum of affine root spaces $\bigoplus \mathbb{C} \varepsilon^k Y_\alpha$
 s.t. $\alpha(\lambda) + k > 0$ $\alpha \in A$

& $\text{Ad}(\ell^{-1})(\varepsilon^k Y_\alpha) \in \mathfrak{g}_\alpha(\mathfrak{g})$

$\iff k - \text{val}(\alpha(\ell)) \geq 0$

$\text{Lie}(\ell G(\mathfrak{g}) \ell^{-1})$
 $= \bigoplus \mathbb{C} \varepsilon^k Y_\alpha$

$\text{Ad}(\ell^{-1})(\varepsilon^k Y_\alpha) \in \mathfrak{g}_\alpha(\mathfrak{g})$

$(\text{Ad}(\ell^{-1}) Y_\alpha = \varepsilon^{-\text{val}(\alpha(\ell))} Y_\alpha)$

\mathfrak{k} is $\tilde{T}(\mathfrak{g})$ -invariant complement \Rightarrow exponential map takes it isomorphically to $\text{Ih}_{\mathfrak{g}}$.

I^+ is pro-unipotent group

$\text{Lie } I^+ \xrightarrow{\sim} I^+$ is isomorphism w/ inverse given by logarithm.

\cup
 $\text{Lie } \text{Stab} \xrightarrow{\sim} \text{Stab}.$

Fixed points

$X^{\mathbb{C}^\times} = G(\mathbb{C}) \Delta G(\mathfrak{g}) / G(\mathfrak{g})$
 fixed points of loop rotation

$X^{\tilde{T}(\mathfrak{g})} = \lambda x_0$
 torsion fixed points

Lemma (1-dimensional orbits of $T(\mathfrak{g})$)

$$X_1 = \bigcup_{\alpha \in \phi^+} X^\alpha$$

If $\alpha \neq \beta \in \phi^+$, $X^\alpha \cap X^\beta = \Lambda$

Proof: • $\dim T = 1 \Rightarrow$ a single root (or none) \leadsto both sides coincide w/ X .

• Assume $\dim T \geq 2$

$T(\mathfrak{g}) \supset H_\alpha$ factors through 1-dim quotient

$$T(\mathfrak{g}) / \ker(\alpha) \Rightarrow X^\alpha \subset X_1.$$

$$X_1 \subset \bigcup_{\alpha \in \phi^+} X^\alpha \iff \left[X_1 \cap \mathfrak{C}_\ell \subset \bigcup_{\alpha \in \phi^+} X^\alpha \cap \mathfrak{C}_\ell \right] \forall \ell.$$

\rightarrow determine 1-dim orbits on \mathfrak{D}_ℓ \rightarrow coordinate axes $\mathfrak{C} \varepsilon^k Y_\alpha$.

$\alpha \in \phi^+$ fixed.

$$\mathfrak{D}_{\ell, \alpha} = \bigoplus_{\substack{k \in \mathbb{Z} \\ \alpha(a) + k > 0 \ \forall a \in \mathfrak{C}_0 \\ \text{val}(\alpha(\ell)) + k < 0}} \mathfrak{C} \varepsilon^k Y_\alpha = \varepsilon^k \text{Lie}(Y_\alpha)(\mathfrak{C}).$$

$C_{\ell, \alpha}$ corresponding subset of the Bruhat cell.

$$C_{\ell, \alpha} = X^{\alpha} \cap C_{\ell} = \text{Bruhat cell of } X^{\alpha}$$

$$\left[I^+ \ell G(\mathfrak{g}) / \mathfrak{b}(\mathfrak{g}) \supset I_2^+ \ell H_{\alpha}(\mathfrak{g}) / H(\mathfrak{g}) \right]$$

If $\alpha \neq \beta$, $D_{\ell, \alpha} \cap D_{\ell, \beta} = \{0\}$ so $X^{\alpha} \cap X^{\beta} = \Lambda$

$$\rightarrow X^{\alpha} = \bigsqcup X^{\alpha} \cap C_{\ell}$$

$$X^{\beta} = \bigsqcup X^{\beta} \cap C_{\ell} \quad \blacksquare$$

SL(2) ASF

$$G(\mathbb{C}) = \text{SL}(2, \mathbb{C}) \quad X^{\text{SL}(2)} = G(\mathbb{C}) / G(\mathfrak{g}) \ni \alpha_0 = \kappa.$$

$$T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad \alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^{\vee} \quad \text{pw-root.}$$

$X^*(T)$

$\alpha^{\vee} : F^{\times} \rightarrow T(F)$ corresponding coroot:

$$\alpha^{\vee}(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

$$\Lambda^{SL(2)} = \langle \alpha^V(\varepsilon) \rangle \quad \ell_m = \alpha^V(\varepsilon^m)$$

$$\parallel$$

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$$

$n \leq -1$

$$x_n = \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} K \in X^{SL(2)}.$$

$\alpha': \mathfrak{t}(F) \rightarrow F$ differential of $\alpha: T(F) \rightarrow F$.

Lemma (Nadler)

$$X^{SL(2)} = \bigsqcup_{n \leq 0} \underbrace{T(F) x_n}_{\text{cplx dimension } |n|}$$

$$\gamma \in \mathfrak{t}(\mathfrak{g}), \quad r = \text{rad}(\alpha'(\gamma)),$$

$$X_\gamma^{SL(2)} = \bigcup_{n=-r}^0 T(F) \cdot x_n \quad \text{union of } T(F)\text{-orbits.}$$

It is preserved by loop rotations.

We let $X_\gamma^{SL(2)} := X_{\leq r}^{SL(2)}$.

Proof: $x \in X^{SL(2)}$ is gK for $g = \begin{pmatrix} \varepsilon^m & b\varepsilon^n \\ 0 & \varepsilon^{-m} \end{pmatrix}$

where either: (1) $b_0 = 0$
 (2) $b_0 \in G^X$ and $n-m < 0$.

$$x_0 = K = G(O)$$

$$B(F) \curvearrowright X = G(F)/G(O)$$

transitively

$$x = gx_0, \quad g = \begin{pmatrix} a & b' \\ 0 & a^{-1} \end{pmatrix} \quad a, b' \in F$$

$$a = a_0 \varepsilon^m \quad a_0 \in G^X$$

right mult. by $\alpha^v(a_0^{-1}) \in K$ get $g = \begin{pmatrix} \varepsilon^m & b \\ 0 & \varepsilon^{-m} \end{pmatrix}$

• $b = 0$ or $\text{val}(b) < m$ ✓

• otherwise, set $b = b_0 \varepsilon^n \quad n \geq m, \quad b_0 \in G^X$

right mult g by $\begin{pmatrix} 1 & -b_0 \varepsilon^{n-m} \\ 0 & 1 \end{pmatrix}$.

• let $x = gK, \quad g = \begin{pmatrix} \varepsilon^m & b_0 \varepsilon^n \\ 0 & \varepsilon^{-m} \end{pmatrix}$.

* If $b_0 = 0, \quad x = \alpha^v(\varepsilon^m) x_0 \in T(F) \cdot x_0$.

* If $b_0 \in G^X, \quad m > n, \quad$ let $a \in G^X$ be a square-root of b_0 .

$$t = \alpha^V(a \varepsilon^m) \in T(F)$$

$$k = \alpha^V(a^{-1}) \in K.$$

$$\begin{aligned} t x_{n-m} k &= \begin{pmatrix} a \varepsilon^m & 0 \\ 0 & a^{-1} \varepsilon^{-m} \end{pmatrix} \begin{pmatrix} 1 & \varepsilon^{n-m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} a \varepsilon^m & a \varepsilon^n \\ 0 & a^{-1} \varepsilon^{-m} \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \\ &= g \end{aligned}$$

dimension statement

$$\text{stabilizer of } x_n = \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} K$$

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} x_n = \begin{pmatrix} a & a \varepsilon^n \\ 0 & a^{-1} \end{pmatrix} K$$

$$\stackrel{?}{=} \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} K$$

$$\Leftrightarrow \begin{pmatrix} 1 & -\varepsilon^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & a \varepsilon^n \\ 0 & a^{-1} \end{pmatrix} \in K$$

$$\begin{pmatrix} a & \varepsilon^n (a - a^{-1}) \\ 0 & a^{-1} \end{pmatrix} \in K$$

$$n \leq 0$$

$$\bullet a \in \mathbb{C}^\times$$

$$\bullet \text{val}(a - a^{-1}) \geq -n$$

$$\gamma \in \mathfrak{t}(\mathfrak{o}) \quad X_\gamma^{SL(2)} = \left\{ g^k \mid g^{-1} \gamma g \in \mathfrak{g}(\mathfrak{o}) \right\}$$

is preserved by $T(F)$.

$$\bullet \gamma = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in \mathfrak{t}(\mathfrak{o}).$$

$$\text{Ad}(x_n) \gamma = \begin{pmatrix} a & -2a\varepsilon^n \\ 0 & -a \end{pmatrix} \in \mathfrak{g}(\mathfrak{o}) \Leftrightarrow \text{val}(a) + n \geq 0$$

turning torus:

$$\begin{pmatrix} 1 & a\varepsilon^n \\ 0 & 1 \end{pmatrix}$$

$$a \in \mathbb{C}^\times$$

$$a = b^2$$

$$\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix}.$$

• dimension $T(F) \cdot x_n$

$$\begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & a\varepsilon^n \\ 0 & a^{-1} \end{pmatrix} \mathcal{K} = \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} \mathcal{K}$$

$$\begin{pmatrix} a & \varepsilon^n(a - a^{-1}) \\ 0 & a^{-1} \end{pmatrix} \in G(\mathcal{G})$$

$$a \in \mathcal{O}^\times$$

$$T(F)$$

$$\& \text{val}(a - a^{-1}) \geq -w$$

$$t(F) \rightarrow T_{x_n} X^{SL(2)}$$

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mapsto \begin{pmatrix} a & a\varepsilon^n \\ 0 & -a \end{pmatrix}$$

$G \subset X^{SL(2)}$ 1-dim orbit, contains two fixed points
 $l_s, l_t \in \Lambda = T(F)/T(G)$ in its closure.

Lemma, $\forall l_s, l_t \in \Lambda$, $\exists!$ 1-dim orbit G_{st} of $\tilde{T}(C)$
in $X^{SL(2)}$ which connects them.

The stabilizer of any point in G_{st} is the kernel of the
affine root $(\alpha, s+t)$.

$$G_{st} \subset X_{\delta}^{SL(2)} \Leftrightarrow \text{val}(\alpha'(\delta)) \geq |s-t|.$$

Proof: \mathcal{T} $T(o)$ -orbit

$$\exists! s \in \mathbb{Z}, \exists! n \leq 0, l_s \alpha_n \in \mathcal{T}$$

$$T(F)/T(G) \cong \mathbb{Z}.$$

$$\mathcal{T} \subset T(F) \alpha_n$$

$$T(F) = \bigsqcup_{s \in \mathbb{Z}} T(o) l_s$$

We prove: $E = s + n$

- (1) the $\tilde{T}(\mathbb{C})$ -orbit G of $l_s \alpha_n$ is 1-dim
- (2) this is the only 1-dim orbit of $\tilde{T}(\mathbb{C})$ in T
- (3) G connects l_s and $l_t = l_s + n\alpha^v(\mathbb{C})$
- (4) $\tilde{T}(\mathbb{C})$ acts on G through the affine root $(\alpha, 2s+n)$.

$\tilde{T}(\mathbb{C})$ -action on the $T(\mathfrak{g})$ -orbit of $l_s \alpha_n$.

$$\alpha^v(b) \in T(\mathfrak{g})$$

$$b = \sum_{i \geq 0} b_i \varepsilon^i$$

$$\alpha \in \mathbb{C}^x \cdot \begin{pmatrix} \varepsilon^s & \varepsilon^{s+n} \\ 0 & \varepsilon^{-s} \end{pmatrix}$$

$(\alpha^v(a), \lambda) \in \tilde{T}(\mathbb{C}) \rightarrow \alpha^v(b) l_s \alpha_n$ is the point

$y = \text{TB Finished.}$

Equivariant homology of $SL(2)$ -Springer fibers

$\mathcal{D} = \mathcal{D}(E) =$ diff ops on E^*

$\cong H_{\tau}^*(pt)$ via Chern-Weil.

$S(E)$ symm algebra of \mathbb{C} -valued ^{val} fcts on E^*

character $\alpha \rightsquigarrow \mathcal{D}_{\alpha} \in \mathcal{D}(E)$ diff op

$S(E) \setminus \{ \mathcal{D}_{\alpha}^d \}$ kernel of \mathcal{D}_{α}^d

Chern-Weil : $H_*^T(\Lambda) \cong \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} S(E)$ of \mathcal{D} -modules.

$\gamma \in \mathcal{H}(G)$

Proposition : $\nu = \text{val}(\alpha'(\gamma))$. The inclusion $\Lambda \subset X$

induces exact sequences:

$$H_*^{\tilde{T}}(X_{\gamma}, \Lambda) \xrightarrow{\tilde{\beta}} H_*^{\tilde{T}}(\Lambda) \rightarrow H_*^{\tilde{T}}(X_{\gamma}) \rightarrow 0$$

\cup \cup \cup

$$H_*^T(X_{\gamma}, \Lambda) \xrightarrow{\beta} H_*^T(\Lambda) \rightarrow H_*^T(X_{\gamma}) \rightarrow 0$$

and the image of β is the submodule \mathcal{D}

$$\sum_{d=1}^v (1-\alpha^v)^d \mathbb{C}[1] \otimes S(t) \{ \partial_{\alpha}^d \} \subset \mathbb{C}[1] \otimes S(t).$$

The coroot α^\vee determines canonical isomorphisms:

$$S(\tilde{E}) \cong \mathbb{C}[\alpha, t]$$

$$\mathcal{D}(\tilde{E}) \cong \mathcal{D}(\partial_{\alpha}, \partial_t)$$

$\forall a, b \in \mathbb{Z}$ s.t. $|b-a| \leq v = \text{val}(\alpha^\vee(\gamma))$, $\exists!$ 1-dim orbit $E_{a,b} \subset X_{\gamma}$ connecting \mathfrak{h}_a and \mathfrak{h}_b and $\tilde{\Gamma}$ acts through the character $\phi_{a,b}$ whose differential

$$\phi_{a,b} : \tilde{E} \rightarrow \mathbb{C} \quad \in E^*$$

corresponds to the differential operator

$$\begin{aligned} \partial_{a,b} &= \partial_{\alpha} + (a+b) \partial_t \\ &= 2\partial_x + (a+b) \partial_t \end{aligned}$$

$$S(m_{a,b}) = S(\ker \phi_{a,b}) \subset S(\tilde{E})$$

$$f_{a,b}^* S(m_{a,b}) = \text{pol fcts on } \tilde{E}^* \text{ annihilated by } \partial_{a,b}.$$

$h(x,t)$ such a function.

$$2\partial_x h(x,t) + (a+b)\partial_t h(x,t) = 0$$

$$g((a+b)x - 2t)$$

$$2(a+b)g'(\dots) + (a+b)(-2)g'(\dots) = 0$$

* § 12.6

* Proposition 12.7

Groups of semisimple rank one

Lemma: If connected reductive cpx lin alg group of rk n and

SS rank 1. Then either

(1) $(\mathbb{C}^*)^{n-1} \times \mathrm{SL}(2, \mathbb{C})$

(2) $(\mathbb{C}^*)^{n-1} \times \mathrm{PGL}(2, \mathbb{C})$

(3) $(\mathbb{C}^*)^{n-2} \times \mathrm{GL}(2, \mathbb{C})$

Lemma:

$$(1) X^H = \bigsqcup_{\ell \in \Lambda^H / \langle \alpha^r \rangle} \ell \phi_\alpha (X^{SL(2)})$$

(2) $T^H(G)$ -orbits on X^H coincide w/ $T^{SL(2)}$ -orbits on X^H .

(3) $\gamma \in E^H(G)$ regular

$$\text{Then } X_\gamma^H = \bigsqcup_{\ell \in \Lambda^H / \langle \alpha^r \rangle} \ell \phi_\alpha (X_{\leq r}^{SL(2)}).$$

$$r = \text{ord}(\alpha'(\gamma)).$$

$$\begin{aligned} & (1 + a_1 t + a_2 t^2 + \dots)^2 \\ &= \mathbf{1} + 2a_1 t + (a_1^2 + 2a_2) t^2 + \dots \end{aligned}$$

Equivariant homology of ASF

$$L_{\alpha, \gamma} = \sum_{t=1}^{\text{val}(\alpha'(\gamma))} (1-\alpha^t)^d \subset \mathbb{C}[1] \otimes S(t) \{ \alpha_2^d \gamma \}.$$

Thm: Suppose $H_*(X_\gamma; \mathbb{C})$ is pure. Then the inclusion $\Lambda \subset X_\gamma$ induces an exact sequence

$$0 \rightarrow \sum_{\alpha \in \phi^+} L_{\alpha, \gamma} \rightarrow \mathbb{C}[1] \otimes S(t) \rightarrow H_*^{\pi(\mathbb{C})}(X_\gamma) \rightarrow 0$$

Combinatorial lemmas

$d, m \in \mathbb{Z}$, $d \geq 1$, Λ free abelian group of rank one.

gen. $\alpha^v \in \Lambda$ determines $t: \Lambda \cong \mathbb{Z}$. $l_a \in \mathbb{Z}$. Mult by α^v acts as shift operator.

$$\mathbb{Q}[\alpha_1, \alpha_2] \cong \mathbb{Q}[1] \otimes \mathbb{Q}[x, t], \quad \ker \partial_t = \mathbb{Q}[1] \otimes \mathbb{Q}[x].$$

$$\text{fm. } d = \sum_{m \leq a < b \leq m+d} C_{ab} (l_b - l_a) \otimes ((a+b)x - t)^{d-1} \in \mathbb{Q}[1] \otimes_{\mathbb{Q}} \mathbb{Q}[x, t]$$

$$C_{ab} = (-1)^{a-b} \binom{d}{a-m} \binom{d}{b-m}.$$

$\mathcal{J}_\alpha \subset \mathbb{Q}[1] \otimes_{\mathbb{Q}} \mathbb{Q}[x, t]$ span:

$$J_{\nu} = \sum_{d=1}^{\nu} \sum_{m \in \mathbb{Z}} Q_{f_{m,d}}$$

Lemma: $d, m \in \mathbb{Z}, d \geq 1$. Then

$$f_{m,d} = (-1)^d d! (1-x^{\nu})^d m \otimes x^{d-1} \in \ker(\partial_t)$$

If $\nu \geq 1$, $J_{\nu} = \sum_{d=1}^{\nu} (1-x^{\nu})^d \frac{Q[\eta] \otimes Q[x] \{\partial_x^d\}}{Q}$

Proof: \forall polynomial f , (Hall, Combinatorial Theory, 1986)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} f(k) = \begin{cases} 0 & \deg(f) \leq n-1 \\ (-1)^n n! & f(k) = k^n \end{cases}$$

$$\cancel{a-b} + \cancel{d-a-j}$$

$$= b' + a - m$$

$$= \cancel{b} + \cancel{d} - \cancel{a} - \cancel{j} - \cancel{2m} + a$$

Lemma 12.4: $d, h, r \geq 1, r \leq h$

$m \in \mathbb{Z}$

$$(*) \quad g = \sum_{\substack{m \leq a < b \leq m+h \\ b-a \leq r}} (b-a) \otimes G_{ab} ((a+b)x - t)^{d-1}$$

$G_{ab} \in \mathcal{A}, b > a$.

If $\partial_t g = 0$ & $d > r$, then $g = 0$

Proof: (formal) $g_{ab} = G_{ab} ((a+b)x - t)^{d-1}$

Sum $(*)$ is $\sum_{a=m}^{m+h-1} \sum_{\substack{b=a+1 \\ b-a \leq r}}^{\min(m+h, a+r)} G_{ab} ((a+b)x - t)^{d-1}$

or $\sum_{b=m+1}^{m+h} \sum_{a=\max(m, b-r)}^{b-1} G_{ab} ((a+b)x - t)^{d-1}$

$\partial_t g = 0$

$$\sum_{b=\max(m, a-r)}^{a-1} G_{ba} (a+b)^j - \sum_{b=a+1}^{\min(m+h, a+r)} G_{ab} (a+b)^j = 0$$

$0 \leq j \leq d-2$

$$\begin{matrix} (a+b)^0 & \dots & \dots & (a+b)^{d-2} \\ \vdots & & & \end{matrix}$$

12.6 $P_r \subset \mathbb{Q}[1] \otimes_{\mathbb{Q}} \mathbb{Q}[x, t]$ vector space spanned by
 $(b-a) \otimes g_{ab}((a+b)x-t)$

g_{ab} polynomials

$$|b-a| \leq r$$

$$P_r \{ \partial_t \} = P_r \cap \ker(\partial_t).$$

Proposition: Fix $r \geq 1$. Then $P_r \{ \partial_t \} = \mathcal{J}_r$:

$$\begin{aligned} \ker(\partial_t) \cap \sum_{|b-a| \leq r} \mathbb{Q}(b-a) \otimes \mathbb{Q}[(a+b)x-t] \\ = \sum_{d=1}^r (1-x^d) \mathbb{Q}[1] \otimes \mathbb{Q}[x] \{ \partial_x^d \}. \end{aligned}$$

Proof: \supset is easy

\subset $P_r \{ \partial_t \}$ vector subspace of P_r spanned by

$$(b-a) \otimes g_{ab}((a+b)x-t)$$

with $g_{ab} = G_{ab} z^h$ homogeneous of degree h .

$$G_{ab} \in \mathbb{Q}.$$

$$P_{r,h} \{ \partial_t \} = P_{r,h} \cap \ker(\partial_t).$$

Then $P_r \{ \partial_t \} = \sum_{h \geq 0} P_{r,h} \{ \partial_t \}.$

Lemma 12.4 $\Rightarrow P_{r,h}[\partial_t] = 0 \quad \forall h \geq r$

Need to show $P_{r,d-1}[\partial_t] \in J_r \quad \forall d \leq r.$

Both are $\mathbb{Q}[\partial_x]$ -modules

$P_r[\partial_t]$ and J_r

Homology of affine Springer fibers in the affine flag manifold

$$F = \mathbb{C}((\epsilon))$$

$$T \subset B \subset G$$

$$I \subset G(F) \quad \text{Iwahori}$$

$$\text{aff flag manifold } Y = Y^G = G(F)/I.$$

$$\Lambda \subset T(F) \quad \text{lattice of translation}$$

$$\alpha^\vee \in \phi^\vee(\alpha, \bar{\Gamma}) \rightsquigarrow \alpha^\vee(\epsilon) \in \Lambda, \\ \text{corot}$$

$$\tilde{W} = \Lambda \rtimes W.$$

$$Y = \bigsqcup_{W \in \tilde{W}} IwI/I \quad \text{Bruhat decomposition.}$$

Each cell has a unique fixed point of $\pi(\epsilon)$:

$$Y^{\pi(\epsilon)} \cong \tilde{W} \quad \text{comp w/ lattice of translations.}$$

$$\alpha \in \phi^+ \quad \text{root} \rightsquigarrow \omega_\alpha \in W \quad \text{reflection.}$$

$$W_\alpha = \{1, \omega_\alpha\}, \quad H^\alpha \text{ conn. red. grp of ss rk 1 containing } T \text{ and}$$

$$U_\alpha \subset G \quad \text{root subgroup.}$$

$$Y^\alpha \text{ aff flag manifold for } H^\alpha.$$

If $u \in Y$ $T(\mathbb{C})$ fixed point,

$$\phi_u : Y^\alpha \xrightarrow{\cong} H^{\alpha, u} \subset Y$$

Restricts to iso of all Springer fibers:

$$Y_{\leq v}^\alpha \cong Y_\delta \cap H^{\alpha, u}. \quad \forall \gamma \in \pm(\mathfrak{g}), \quad v = \text{rad } \alpha'(\gamma).$$

$$Z^\alpha := \bigcup_{u \in W_\alpha \setminus W} H^{\alpha, u} = \bigsqcup_{u \in W_\alpha \setminus W} \phi_u(Y^\alpha)$$

Lemma: $\gamma \in \pm(\mathfrak{g})$ regular.

The union of 0 & 1-dim orbits of $T(\mathbb{C})$ in the aff Spr. fibers is

$$(Y_\delta)_1 = \bigcup_{\alpha \in \phi^\pm} Z_\gamma^\alpha$$

$$Z_\gamma^\alpha = Y_\delta \cap Z^\alpha = \bigsqcup_{u \in W_\alpha \setminus W} \phi_u(Y_{\leq v}^\alpha)$$

$$\text{If } \alpha \neq \beta, \quad Z_\gamma^\alpha \cap Z_\gamma^\beta = \emptyset.$$

Claim: Each $\alpha \in \phi^+$ corresponds to a deg. one diff operator

$$\partial_\alpha \in \mathcal{D}(t), \alpha^\vee \in \Lambda \text{ and } w_\alpha \in W.$$

$\alpha \in \phi^+$, define the $\mathcal{D}(t)$ submodule $M_{\alpha, \gamma}$ of

$$\mathbb{C}[\tilde{w}] \otimes_{\mathbb{C}} S(t).$$

$$M_{\alpha, \gamma} = \sum_{d=1}^{\text{ord}(\alpha^\vee \gamma)} (1 - \alpha^\vee)^d \mathbb{C}[\tilde{w}] \otimes_{\mathbb{C}} S(t) \{ \partial_\alpha^{d\gamma} \} \\ + \sum_{d=1}^{\text{ord}(\alpha^\vee \gamma)} (1 - \alpha^\vee)^{d-1} (1 - w_\alpha) \mathbb{C}[\tilde{w}] \otimes_{\mathbb{C}} S(t) \{ \partial_\alpha^d \}.$$

$\gamma \in t(\mathfrak{g})$ regular element.

Assume $H_*^\gamma(Y_\gamma; \mathbb{C})$ is pure.

Then, $\tilde{w} \subset Y_\gamma$ induces an exact sequence of $\mathcal{D}(t)$ -modules

$$0 \rightarrow \sum_{\alpha \in \phi^+} M_{\alpha, \gamma} \rightarrow \mathbb{C}[\tilde{w}] \otimes_{\mathbb{C}} S(t) \rightarrow H_*^{\gamma(\mathbb{C})}(Y_\gamma) \rightarrow 0$$

$(\tilde{W} \times \text{Aut})_g$ acts on this eq. hom. grp & restricts to an action on the ordinary homology

$$H_* (Y_r) = H_*^{T(\mathbb{C})} (Y_r) \{I\}$$

Springs action might

$$\tilde{W} \curvearrowright \mathbb{C}[\tilde{W}] \otimes S(E)$$

regular \otimes trivial

It preserves each relation $M_{\alpha, \gamma} \in \phi^+$.

Assume $H_* (Y_r; \mathbb{C})$ is pure -

induces action of $\mathbb{C}[\tilde{W}]$ on homology compatible w/ the $\mathcal{D}(t)$ module structure, & commute w/ the $(\tilde{W} \times \text{Aut})_g$ act.

\leadsto It restricts to action of \tilde{W} on ordinary cohomology, coincides w/ Lusztig action.

action of trigonometric DAHA.

Affine flag manifold for $SL(2)$

$$G = SL(2)$$

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$I \subset G(\mathbb{C})$ Iwahori

$Y = Y^{SL(2)} = G(\mathbb{F})/I$ affine flag manifold.

$x_0 = I/I \in Y$ base point.

$T \subset G$ torus of diag. matrices

$\Lambda = \{ \alpha^\vee(\varepsilon^m) : m \in \mathbb{Z} \}$ translation lattice

$\alpha^\vee \in \phi^\vee$ simple coroot determined by $T \subset B$.

$T(\mathbb{C}) \curvearrowright Y$ w/ fixed points

$$lm = \begin{pmatrix} \varepsilon^m & 0 \\ 0 & \varepsilon^{-m} \end{pmatrix} I$$

$$\text{and } rm = \begin{pmatrix} 0 & \varepsilon^m \\ -\varepsilon^{-m} & 0 \end{pmatrix} I, \quad m \in \mathbb{Z}$$

$W = \{1, w_2\}$ Weyl group of G .

$\tilde{W} = \Lambda \rtimes W$ acts simply transitively on these fixed points.

$\tilde{W} \cong$ fixed points

* $d^r(\varepsilon^m) \mapsto l_m$

* $\alpha_{\alpha} \mapsto r_0 \in Y$

) so $\begin{cases} d_a d_b = l_{a+b} \\ l_a r_b = r_{a+b} \\ r_a d_b = r_{a-b} \\ r_a r_b = l_{a-b} \end{cases}$

* $r_0 = l_0$ base point

* $\alpha_{\beta_1} = r_0$

$\tilde{T}(C)$ extended torus.

If $m \leq -1$, $s \in \mathbb{Z}$, the $\tilde{T}(C)$ -orbit of $l_s \alpha_m$ is 1-dim, and connects l_s and r_{m+s} .

$\pi: \begin{matrix} Y^{SL(2)} \\ \downarrow \\ X^{SL(2)} \end{matrix}$ proj to affine grassmannian.

$G(F)$ equiv fibration w/ fibers $G(C)/B(C) \cong \mathbb{P}_C^1$.

$\pi(l_m) = \pi(r_m) \quad \forall m,$

$\pi(\alpha_m) = \pi(\gamma_m) \quad m \leq 0.$

For $m \leq 0$, $T(F) \cdot \alpha_m$ is $(-m)$ -dim projects under π iso to $T(F) \cdot \pi(\alpha_m)$.

For $m \leq 1$, $\dim(T(F) \cdot y_m) = 1 - m$.

If $m \leq 0$, it fibers over the $T(F)$ -orbit of $\pi(y_m) = \pi(x_m)$

as/ 1-dim affine spaces
fibers

• Fix $\gamma \in \mathfrak{t}(\mathfrak{g})$, $Y_\gamma = \{xI \in Y : \text{Ad}(x^{-1})(\gamma) \in \text{Lie}(I)\}$

preserved by $\tilde{T}(\mathfrak{C})$,

$\pi: Y_\gamma \rightarrow X_\gamma$ surj. Not a fibration.

$H_*(Y_\gamma)$ is pure

prop: $Y = \bigsqcup_{n \leq 0} T(F) \cdot \{x_n \cup y_{n+1}\}$

$Y_\gamma = Y_{\leq r} = \bigcup_{n=-r}^0 T(F) \cdot \{x_n \cup y_{n+1}\}$

$r = \text{val}(\alpha'(\gamma))$

$\forall (s, t) \in \mathbb{R}^2$, $\exists!$ $Est \subset Y$ 1-dim orbit of $\tilde{T}(\mathfrak{C})$, connecting I_s and I_t giving all 1-dim orbits of $\tilde{T}(\mathfrak{C})$ in Y .

$\tilde{T}(\mathfrak{C})$ acts on Est through $(\alpha, s+t)$.

$Est \subset Y_\gamma \Leftrightarrow -r \leq t-s \leq r+1$.

• α^v gives $S(\tilde{E}) \cong \mathbb{C}[x, t]$

$$\mathcal{D}(\tilde{E}) \cong \mathbb{C}[\partial_x, \partial_t].$$

$\gamma \in E(b)$, $v = \text{val}(\alpha'(\gamma))$.

$\Pi_{ab} \subset \mathbb{C}[\tilde{w}] \otimes_{\mathbb{C}} S(\tilde{E})$ spanned by

$$(l_a - r_b) \otimes g_{ab}((a+b)x - 2t)$$

g_{ab} polynomial.

Π_{ab} is $\mathcal{D}(\tilde{E})$ -module.

$$Q_v = \sum_{-v \leq b-a \leq v-1} \Pi_{ab}$$

$\tilde{w} \in Y_\gamma$ induces s.e.s in $T(\mathbb{C})$ -equiv coh:

$$0 \rightarrow Q_v \{\partial_t\} \rightarrow \mathbb{C}[\tilde{w}] \otimes_{\mathbb{C}} S(\tilde{E}) \rightarrow H_*^{T(\mathbb{C})}(Y_\gamma) \rightarrow 0$$

Proposition: For $v \geq 1$, the module of relations $Q_v \{\partial_t\}$ is

spanned by

$$\sum_{d=1}^v (1 - \alpha^v)^d \mathbb{C}[\tilde{w}] \otimes S(\tilde{E}) \{\partial_x^d\}$$

and
$$\sum_{d=1}^v (1 - \alpha^v)^{d-1} (1 - w_\alpha) \in [\tilde{W}] \otimes S(t) \{ \partial_\alpha^d \}.$$

Next step: groups of semisimple rank one

H ss rank 1.

$$T \subset B \subset H$$

$$I \subset H(F) \supset w.$$

$\Upsilon^H = H(F)/I$ aff flag man. for H .

$\alpha, \alpha^v \rightsquigarrow w_\alpha \in W$ in Weyl grp of H

$$\langle \alpha^v \rangle \subset \Lambda^H$$

1-dim.

$\partial_\alpha \in \mathcal{D}(t)$ degree 1.

$\tilde{W} = \Lambda \rtimes W$. extended affine Weyl grp.

$SL(2) \hookrightarrow H$ gives

$$\phi_\alpha : \Upsilon^{SL(2)} \rightarrow \Upsilon^H$$

$\Upsilon \in \mathcal{E}(0)$ reg.

$v = \text{val } \alpha^v(\Upsilon)$.

$$\Upsilon^H = \bigcup_{\ell \in \Lambda^H / \langle \alpha^v \rangle} \ell \cdot \phi_\alpha (\Upsilon^{SL(2)})$$

$$Y^H = \sqcup_{\ell \in \Lambda^H / \langle \alpha^v \rangle} \ell \phi_\alpha (Y_{\leq v}^{S_2(\alpha)})$$

$$\begin{matrix} \Pi \\ \vdots \\ Y^H \\ Y_{\leq v} \end{matrix}$$

Prp. We have

$$0 \rightarrow \mathcal{O}_v \{ \mathcal{Z}_z \} \rightarrow \mathbb{C}[\tilde{w}] \otimes_{\mathbb{C}} S(\mathbb{C}) \rightarrow H_{\mathbb{C}}^{\Pi(\alpha)} (Y_{\leq v}^H) \rightarrow 0$$

Hessenberg varieties (De Mari, Procesi, Shayman)

$$h: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$$

Hessenberg function: $h(i+1) \geq \max(i, h(i))$

$$X: \mathbb{C}^n \rightarrow \mathbb{C}^n$$

$$X(X, h) := \left\{ F \text{ full flag} \mid X \cdot F_i \subset F_{h(i)} \right\}.$$

- generalized version of this -