Affine Springer theory & representations of DAHA

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(based on Varagnolo-Vasserot, Oblomkov-Yun, general considerations)

① Representations of affine Hecke algebras

Usual Hecke algebra $= (\underbrace{\mathrm{Fun}}_{\mathbb{C}\text{-valued}}(B\backslash \underbrace{G}_{\text{everything is over } \mathbb{F}_q}/B), \overset{\text{conv. product}}{*})$

We can do a series of "transformations":

$$\mathrm{Fun}(B\backslash G/B) \underset{\text{fct}\leftrightarrow\text{sheaf}}{\rightsquigarrow} \mathrm{Sh}(B\backslash G/B) \underset{\text{Riemann-Hilbert}}{\rightsquigarrow} \mathcal{D}\text{-mod}(B\backslash G/B) \underset{\text{classical limit}}{\rightsquigarrow} \mathrm{Coh}^{\mathbb{C}^*}(T^*[B\backslash G/B]) \rightsquigarrow K^{\mathbb{C}^*}(T^*(B\backslash G/B)),\ \rightsquigarrow H_*^{\mathbb{C}^*}(T^*(B\backslash G/B))$$

What is $T^*(B\backslash G/B)$? $\mu^{-1}(0)$ or union of conormal bundles to B-orbits

$$T^*(B\backslash G/B) = T^*[(G/B)/B] = [T^*_B(G/B)/B] = \left\{ \begin{array}{l}\text{Conormal at } B' \subset G/B \text{ is} \\ (\mathfrak{g}/\mathfrak{b}+\mathfrak{b}')^* \hookrightarrow (\mathfrak{g}/\mathfrak{b}')^*,\ \text{i.e. } (\mathfrak{n}'\cap\mathfrak{n}) \subset \mathfrak{n}' \end{array}\right\} \qquad \mathfrak{n} \subset \mathfrak{g} \text{ nilradical}$$

$$= [(\underbrace{G/B\times\mathfrak{n}}_{\text{inside } G/B\times\mathfrak{g}}) \cap (G\times_B \mathfrak{n})/B] \underset{\mathrm{Ind}_B^G}{=} [\underbrace{(G\times_B \mathfrak{n}_1)\cap(G\times_B\mathfrak{n}_2)}_{\text{inside } G/B\times G/B\times\mathfrak{g}}/G] = [\tilde{\mathcal{N}}\times_{\mathcal{N}}\tilde{\mathcal{N}}/G],\ \text{where } \begin{array}{c}\tilde{\mathcal{N}}\\ \downarrow\\ \mathcal{N}\end{array} \text{ is Springer resolution.}$$

We've seen that $K^{G\times\mathbb{C}^*}(\tilde{\mathcal{N}}\times_{\mathcal{N}}\tilde{\mathcal{N}}) =$ affine Hecke algebra
$H^{G\times\mathbb{C}^*}(\underbrace{\tilde{\mathcal{N}}\times_{\mathcal{N}}\tilde{\mathcal{N}}}_{Z}) =$ deg. affine Hecke.

So fts $\rightsquigarrow$ Coh (T^*) buys us affinization.

What about fin.dim reps? $a \in G\times\mathbb{C}^*$ semisimple

$$\mathbb{C}_a \otimes_{K^{G\times\mathbb{C}^*}(pt)} K^{G\times\mathbb{C}^*}(Z) \underset{\text{equiv. formality}}{\simeq} \mathbb{C}_a \otimes_{\langle a\rangle} K^{\langle a\rangle}(Z) \simeq \mathbb{C}_a \otimes K^{\langle a\rangle}(Z^{\langle a\rangle}) \simeq K(Z^a)$$

One can show that LHS corr. to central quotients of AHA.

comp. group

$$K(Z^a) \curvearrowright K(\tilde{\mathcal{N}}^a_x) \curvearrowleft C(a,x)^\circ$$

has to be preserved by a

simult. centralizer of a, x

$\rightsquigarrow$ fin.dim. rep $L_{a,x,\chi}$ of AHA

$\chi \in \mathrm{Irr}\, C(a,x)^\circ$

Thm (Kazhdan-Lusztig, Chriss-Ginzburg, Lusztig)

This recovers all ~~rep~~ f.d. reps of AHA.

(q is not a root of unity).

② Second affinization

Try to repeat the same story in the affine situation.

1) $I = B \cdot tG[[t]] \quad \subset G((t))$

2) $\mathrm{Fun}(I \backslash G((t)) / I)$ has conv. product, recovers affine Hecke.

3) We are drawn to consider $K(T^*(I \backslash G((t)) / I))$ what is this??

$$T^*(I \backslash G((t))/I) \simeq \underbrace{(G((t))/I \times \dot{n}) \cap (G((t)) \times_I \dot{n})}_{\Lambda} / I$$

take this as definition

$\dot{n} = n + t\mathfrak{g}[[t]]$

What is $K^I(\Lambda)$?

- $G((t))/I$ is a union of fin.dim. I-orbits $\rightsquigarrow \Lambda = \bigcup_{\mathcal{O}} \Lambda_{\leq \mathcal{O}}$ closures of orbits
- $\dot{n}_{\leq \mathcal{O}}$ over $\Lambda_{\leq \mathcal{O}}$ is a pro-vector bundle : $\dot{n}_{\leq \mathcal{O}} = \varprojlim \dot{n}^{(i)}_{\leq \mathcal{O}}$
- $\Lambda = \varinjlim \varprojlim \Lambda^i_{\leq \mathcal{O}}$ $\rightsquigarrow$ K-theory is $K(\varinjlim \varprojlim \mathrm{Coh}(\Lambda^i_{\leq \mathcal{O}}))$

Thm (Varagnolo-Vasserot, ...)

$$K^{I \times \mathbb{C}^*}(\Lambda) \simeq \mathrm{DAHA}_G$$

$$H^{I \times \mathbb{C}^*}(\Lambda) \simeq \text{trigonometric DAHA}$$

③ Flavours of DAHA

G - semisimple Lie group, $n = \mathrm{rk}\, G$, P - weight lattice, Q - root lattice

a) "full" DAHA : $\mathbb{C}_{q,t}$ algebra $\mathbb{H}$

generated by $\underbrace{T_0, T_1, \dots, T_n}_{\text{affine Hecke alg}}$, $\underbrace{\Omega}_{P^\vee/Q^\vee}$, $X^{P \times \mathbb{Z}}$; set $X^{(\lambda, i)} := q^i X^\lambda$

relations : • affine Hecke relations for T_i's (def. parameter t)

• $\forall \pi_r \in \Omega$ $\quad \pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j$

$\pi_r X^\mu \pi_r^{-1} = X^{\pi_r(\mu)}$

• $T_i X^\mu = X^\mu T_i$ if $\langle \mu, \alpha_i^\vee \rangle = 0$ $\quad$ where $\alpha_0^\vee = -\theta^\vee$ (highest root)

$T_i X^\mu = X^{s_i(\mu)} T_i^{-1}$ if $\langle \mu, \alpha_i^\vee \rangle = 1$.

$\underline{\text{Size}}$: $\underbrace{\mathbb{C}[P]}_{Y\text{'s}} \otimes \underbrace{\mathbb{C}[Q^\vee] \otimes \mathbb{C}[W]}_{H^{\text{aff}}} \otimes \mathbb{C}[\Omega]$ $\quad$ $\mathbb{C}[P^\vee]$ $\quad \rightsquigarrow \quad \mathbb{C}[W] \otimes \mathbb{C}[P] \otimes \mathbb{C}[P^\vee]$.

b) trig DAHA

$\mathbb{H}^{\text{trig}}$: $\mathbb{C}_{c,t}$ -alg. gen. by $W^{\text{aff}} \rtimes \Omega$ (extended aff Weyl $W \ltimes P^\vee$), $\mathbb{C}[\mathfrak{h}] = \mathbb{C}[y_1, \dots, y_n]$, $y_\lambda = \sum \langle \lambda, \alpha_i \rangle y_{\kappa_i}$

$\underline{\text{relations}}$: $s_i y_\lambda - y_{s_i(\lambda)} s_i = -c \langle \lambda, \alpha_i \rangle u \quad i = 1, \dots, n$

$s_0 y_\lambda - y_{s_0(\lambda)} s_0 = c \langle \lambda, \theta \rangle u$

action of W : $\lambda \cdot y_\mu = y_\mu - \langle \lambda, \mu \rangle t$.

$\pi_r y_\lambda = y_{\pi_r(\lambda)} \pi_r$

$\underline{\text{Size}}$ $\quad \mathbb{C}[P^\vee] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}]$

$\underline{\text{Rmk}}$ $\mathbb{H}^{\text{trig}}$ is filtered, not graded! Can be made graded by introducing auxillary u

c) rat DAHA

gen. by $\mathbb{C}[\mathfrak{h}]$, $\mathbb{C}[\mathfrak{h}^\vee]$, W $\qquad x \in \mathfrak{h}, y \in \mathfrak{h}^\vee$

rel: $\quad wx = w(x) w \qquad [y, x] = t \langle y, x \rangle - \sum_{\alpha \in \Delta^+} c \langle y, \alpha \rangle \langle \alpha^\vee, x \rangle s_\alpha$.

$wy = w(y) w$

Relation to Vasserot-Vasserot

$$K^I(pt) = K^T(pt) = \mathbb{C}[P]$$
$$\downarrow$$
$$K^I(\Lambda)$$

Affine Hecke alg. generators corr. to structure sheaves of irred. comp. of Λ corr. to small I-orbits.

$K \rightsquigarrow H \quad : \quad \mathbb{H} \rightsquigarrow \mathbb{H}^{trig}$

④ **Geometric representations**

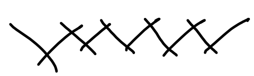
$K^{I \times \mathbb{C}^\times_{rot} \times \mathbb{C}^\times_{dil}}$

$$\mathrm{DAHA} \curvearrowright K(\underbrace{T^*(G((t))/I)}_{=\ \{(x, I') \in \mathfrak{g}((t)) \times G((t))/I \ :\ x \in n(I')\} =: \dot{\mathcal{N}}})$$

$\forall\ x \in \mathfrak{g}((t))^{\text{loc. nilp}}$, $a \in G((t))$ semisimple, we expect $\mathrm{DAHA} \curvearrowright K(\dot{\mathcal{N}}_x^a)$

$\dot{\mathcal{N}}_x$ - affine Springer fiber

Problems:
1) Aff. Sp. fibers can be infinite dimensional!
   Kazhdan-Lusztig: ok if x is regular semisimple OR $a \in T^{reg}$ (?)
2) we shouldn't expect DAHA to have f.d. reps in general!

   Ex $G = SL_2$ $\quad \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \rightsquigarrow$ 

   $\begin{pmatrix} 0 & 1 \\ t^3 & 0 \end{pmatrix} \rightsquigarrow \mathbb{P}^1$

Affine Springer fibers are projective $\iff$ x is elliptic regular semisimple.

Def $x \in \mathfrak{g}((t))^{rs}$, $\quad p(x) \in (\mathfrak{g}/G)((t))$ - char. polynomial.

We have two actions $\mathbb{G}_m \curvearrowright (\mathfrak{g}/G)((t))$:
- loop rotation $\quad (t \mapsto \lambda t)$
- scaling $\mathfrak{g}$ $\quad$ (acts on $\mathfrak{g}/G$ by exponents of G)

x is homogeneous of slope d/m if $p(x)$ is fixed scaling$^d\cdot$rot^{-m}

x is elliptic if m is an elliptic regular number of W E.g. $W = S_n$ ell. reg. $= \{n+1\}$.

Problem 3 We don't know how to produce reps of $\mathbb{H}^{\mathrm{rat}}$ yet.
But, algebraically: $M \in \mathbb{H}^{\mathrm{trig}} \rightsquigarrow$ filtration s.t. X's act unipotently $\rightsquigarrow$ $x_i = 1 - X_i$ gives $\mathbb{H}^{\mathrm{rat}} \curvearrowright \mathrm{gr}\, M$.

Oblomkov–Yun: construct this filtration geometrically.

⑤ Global setting

Recall that affine Grassmanians appear when considering $\mathrm{Bun}_G C$:

$$\begin{array}{ccccc} \mathrm{Hecke} & \longrightarrow & \mathrm{Bun}_G C & \longrightarrow & BG[[t]] = \mathrm{Bun}_{\mathbb{D}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bun}_G C & \longrightarrow & \mathrm{Bun}_G(C\backslash c) & \longrightarrow & BG((t)) = \mathrm{Bun}_{\mathbb{D}^\circ} \end{array}$$

$\rightsquigarrow$ fiber of Hecke $\to$ Bun is $G((t))/G[[t]]$

of Hecke $\to \mathrm{Bun}_G(C\backslash c)$ is $G[[t]]\backslash G((t))/G[[t]]$

Similarly; $I \backslash G((t)) / I \curvearrowright$ G-bundles on C with B-reduction at c.

parabolic Higgs bundles.

$\rightsquigarrow$ DAHA $\curvearrowright K(T^*\mathrm{Bun}_G^{B,c})$

$\downarrow$ $\leftarrow$ compatible with modifications at points $\Rightarrow$ DAHA $\curvearrowright K$(Hitchin fibers)

$\mathcal{A}$ – Hitchin base

We know: $\mathcal{A}^\heartsuit \subset \mathcal{A}$ where fibers are projective varieties.

Problems 1) $\mathfrak{g}$, $\Phi \in H^0(\mathrm{Ad}_{\mathfrak{g}} \otimes \cancel{K_c})$ L, $\deg L \geq 2g$ $\deg(K_c) = 2g-2$

2) If we do just this (Yun) $\rightsquigarrow$ u will act by 0!

Solution (Oblomkov-Yun): consider orbifold curves!

$\mathcal{O}_X(\frac{1}{m})^{\otimes m} = \mathcal{O}_X(\infty)$.

Setup: $C = \mathbb{P}^1$ with orbifold str. at ∞ of order m

Consider G-Higgs bundles on C with B-reduction at 0, $K_{\mathbb{P}^1} \rightsquigarrow L = \mathcal{O}_X(\frac{d}{m})$

Moduli of such G-Higgs bundles has two $\mathbb{G}_m$-actions:

- dilate the Higgs field
- rotate $\mathbb{P}^1$ (preserving $0, \infty$)

$\mathcal{A}^\heartsuit \rightsquigarrow \mathcal{A}^\heartsuit_{d/m}$ - elts preserved by 1-dim torus $\mathrm{dil}^m \cdot \mathrm{rot}^{-d}$.

Thm (OY)

- we can identify Hitchin base $\mathcal{A}^\heartsuit_{d/m}$ with reg. semisimple elts of slope $\frac{d}{m}$.
- Hitchin fiber $\simeq$ [aff Springer fiber $/S$]. ← product formula.

for some meaning of $\simeq$ — "centralizer" group scheme

- $\exists\ \mathbb{H}^{\mathrm{trig,gr}} \curvearrowright H_*(\text{fiber})$, u acts by $-d \cdot c_1(\mathrm{dil}^m \cdot \mathrm{rot}^{-d})$
- $H_\cdot(Sp)^{S \times B_r}$ is fin. dimensional.

⑥ Modules over $\mathbb{H}^{\mathrm{rat}}$

Now that we've reinterpreted $H_\cdot(Sp)^{S\times B_r}$ as $H(\text{Hitchin fiber})$, we can transport perverse filtration!

Thm This filtration is "compatible" with the action of $\mathbb{H}^{\mathrm{trig,gr}}$

Taking $\mathrm{Gr}_P H_\cdot(Sp)^{S\times B_r}$ - get $\mathbb{H}^{\mathrm{rat}}$-modules.

$u \mapsto \frac{d}{m}$ $\rightsquigarrow$ irreducible f.d. spherical module $L_{\frac{d}{m}}(\mathrm{triv})$ ← induced from 1-dim. representation of affine Hecke algebra.